T	=	temperature
x	=	liquid phase mole fraction
y	=	vapor phase mole fraction

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Liquid Film Flowing Slowly Down a Wavy Incline

A viscous lamina of given mean thickness flows down a wavy incline. Assuming low Reynolds number and small perturbations due to the wavy striations, the velocity profiles and the free surface profile are determined. It is found that the amplitude and phase shift of the free surface depend, in a complicated manner, on the surface tension and the wave length and orientation of the wavy striations. The motion of a particle on the free surface experiences drift which is also a function of the surface tension, the amplitude, wave length and orientation of the wavy incline.

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SCOPE

The flow of a liquid lamina down an inclined plane is important in many chemical engineering processes, e.g., film cooling, dip painting, swept film evaporators and vapor condensers. For low Reynolds numbers (less than about 5) the flow is smooth and laminar. Instability or ripples appear on the surface when the Reynolds number is larger than about 3 cosec α, where α is the inclination of the plate. The flow becomes turbulent when the Reynolds number is over 250. An excellent review of the theory and experiments on film flow was given by Fulford (1964).

The low Reynolds number flow down a smooth plate was first investigated by Hopf (1910) and Nusselt (1916). The velocity profile was a simple parabola. No theoretical or experimental work exists when the plate is not smooth but wavy, although some experiments show ridges or surface roughness increase turbulence at high Reynolds numbers (e.g. Laird et al 1962). The present paper studies the flow down a wavy or striated plate at low Reynolds numbers. The striations may be caused by wear, imperfect machining or they may be intentional as on textured surfaces. Particular emphasis is placed on the effect of striations on flow rate and on the lateral drift of the fluid when the direction of the striations is at an angle with the horizontal plane.

CONCLUSIONS AND SIGNIFICANCE

It is found, for a fixed mean depth of the film, the flow transverse to the striations is decreased in comparison to that of a smooth plate while the flow along the striations is increased. The decrease and increase of flow are not compensatory and depend on complicated functions of the wave length and a parameter D which includes the effect of the orientation of the striations and the surface tension. The fluid particles thus experience a drift (a tendency to flow in the direction of the striations). The mean drift angle increases with the amplitude of bottom striations. It decreases with increased wave length, inclination of the plate and surface tension. The striations on the inclined plate thus exert considerable influence on the transport properties of film flow.

The film flow over an inclined wavy plate, especially the three dimensional flow studied in the paper, has never been investigated before. It is hoped that the present work would lead to further discussion on this important topic, both experimentally and theoretically.

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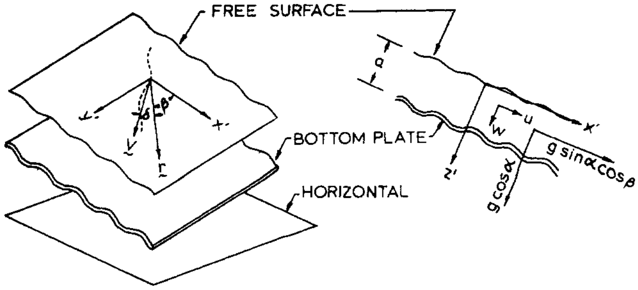

Figure 1. Coordinate system showing particle path (dotted line).

INTRODUCTION

The major difficulty of the problem of fluid flow with a free surface is that the location of the free surface is not known *a priori*. For two dimensional inviscid potential flow, this difficulty can be circumvented by a transformation into the hodograph (velocity) plane. The success of this method relies on the fact that the two dimensional Laplace's equation is invariant under conformal transformation. Viscous free surface flows, unfortunately, do not have this property. Thus, very few solutions exist for viscous flow with a free surface.

Figure 1 shows the coordinate system. Let (x', y', z') be Cartesian coordinates with unit directions ($\underline{i}$, $\underline{j}$, $\underline{k}$) respectively. The wavy plate is situated at $z' = a + b\sin(2\pi x'/l)$ and the still unknown free surface is at $z' = af(x')$. We require that $f(x')$ have zero mean such that the mean distance between the boundaries is constant a. Let $\underline{g}$ be the gravitational acceleration and let α (the mean angle of incline of the plate) and β (the angle of the wavy striations) be defined as follows

$$g\cos\alpha = \underline{g}\cdot\underline{k} \tag{1}$$

$$g\sin\alpha\cos\beta = \underline{g}\cdot\underline{i} \tag{2}$$

$$g\sin\alpha\sin\beta = \underline{g}\cdot\underline{j} \tag{3}$$

$$g \equiv |\underline{g}| \tag{4}$$

without loss of generality, we restrict to the range $0 < \alpha < \pi/2$, $0 \le \beta \le \pi/2$. It is evident from the geometry that pressure, velocity and the shape of the free surface are independent of y'. The governing equations are:

$$u'u'_{x'} + w'u'_{z'} = -\frac{1}{\rho}\frac{\partial p'}{\partial x'} + \nu(u'_{x'x'} + u'_{z'z'}) + g\sin\alpha\cos\beta \tag{5}$$

$$u'v'_{x'} + w'v'_{z'} = \nu(v'_{x'x'} + v'_{z'z'}) + g\sin\alpha\sin\beta \tag{6}$$

$$u'w'_{x'} + w'w'_{z'} = -\frac{1}{\rho}\frac{\partial p'}{\partial z'} + \nu(w'_{x'x'} + w'_{z'z'}) + g\cos\alpha \tag{7}$$

$$\frac{\partial u'}{\partial x'} + \frac{\partial w'}{\partial z'} = 0 \tag{8}$$

The boundary conditions are:

$$\text{on } z' = a+b\sin(2\pi x'/l), \quad u' = v' = w' = 0, \tag{9}$$

and on the unknown free surface $z' = af(x')$

$$\text{shear stresses} = 0 \tag{10}$$

$$\text{normal stress} = -p'_a - \frac{taf_{x'x'}}{[1 + a^2(f_{x'})^2]^{3/2}} \tag{11}$$

We first solve Eqs. 5, 7 and 8 for the flow across striations (in $\underline{i}$ direction), then use these results to solve Eq. 6 for the flow along striations (in $\underline{j}$ direction).

Flow Across Striations

Let Q be the (still unknown) amount of volume flow rate in the $\underline{i}$ direction. We normalize all lengths by a, the velocities by Q/a, the pressure by $\rho\nu Q/a^2$ and drop primes. Eqs. 5-8 become:

$$R(uu_x + wu_z) = -\frac{\partial p}{\partial x} + u_{xx} + u_{zz} + g\sin\alpha\cos\beta a^3/(\nu Q) \tag{12}$$

$$R(uu_x + wu_z) = v_{xx} + v_{zz} + g\sin\alpha\sin\beta a^3/(\nu Q) \tag{13}$$

$$R(uw_x + ww_z) = -\frac{\partial p}{\partial z} + w_{xx} + w_{zz} + g\cos\alpha a^3/(\nu Q) \tag{14}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{15}$$

For the flow across striations, we define a stream function:

$$\psi = \int u dz = -\int w dx \tag{16}$$

Then Eqs. 12 and 14 reduce to:

$$R[\psi_z\nabla^2\psi_x - \psi_x\nabla^2\psi_z] = \nabla^4\psi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)^2\psi \tag{17}$$

We now assume the amplitude of the wavy plate is small compared to the depth of the fluid, while the wave length is of same order as the depth:

$$\epsilon \equiv \frac{b}{a} \ll 1, \quad \lambda = 2\pi a/l \approx 0(1) \tag{18}$$

We also assume the Reynolds number is of order ϵ or smaller:

$$R = \kappa\epsilon, \quad \kappa = 0(1) \tag{19}$$

This is reasonable for fluids of high viscosity. Let s be the direction tangent to the boundary and n be the direction normal to the boundary. The boundary conditions become:

$$\text{on } z = 1 + \epsilon\sin\lambda x, \ \psi = 1, \ \frac{\partial\psi}{\partial n} = 0 \tag{20}$$

$$\text{on } z = f(x), \quad \psi = 0 \tag{21}$$

$$T_{ns} = 0, \quad T_{nn} = -p_a - \frac{\Lambda f_{xx}}{(1 + f_x^2)^{3/2}} \tag{22}$$

The boundary conditions (Eqs. 20-22) are extremely difficult to apply directly because $f(x)$ is unknown. Since ϵ is small, the perturbed surface $f(x)$ should also be small. We expand the variables with respect to ϵ:

$$\psi(x, z) = \psi_0 + \epsilon\psi_1 + \epsilon^2\psi_2 + \ldots \tag{23}$$

$$p(x, z) = p_0 + \epsilon p_1 + \epsilon^2 p_2 + \ldots \tag{24}$$

$$f(x) = \epsilon g_1 + \epsilon^2 g_2 + \ldots \tag{25}$$

$$Q = Q_0 + \epsilon Q_1 + \epsilon^2 Q_2 + \ldots \tag{26}$$

Eq. 20 yields:

$$\begin{aligned} 1 = \psi|_{1+\epsilon\sin\lambda x} &= (\psi_0 + \epsilon\psi_1 + \epsilon^2\psi_2 + \ldots)|_{1+\epsilon\sin\lambda x} \\ &= \psi_0|_1 + \epsilon\sin\lambda x\psi_{0z}|_1 + \frac{\epsilon^2}{2}\sin^2\lambda x\psi_{0zz}|_1 \\ &\quad + \epsilon\psi_1|_1 + \epsilon^2\sin\lambda x\psi_{1z}|_1 + \epsilon^2\psi_2|_1 + \ldots \end{aligned} \tag{27}$$

$$\begin{aligned} 0 = \frac{\partial\psi}{\partial n}\Big|_{1+\epsilon\sin\lambda x} &= \nabla\psi\cdot\frac{\nabla(z - 1 - \epsilon\sin\lambda x)}{|\nabla(z - 1 - \epsilon\sin\lambda x)|}\Big|_{1+\epsilon\sin\lambda x} \\ &= \frac{1}{\sqrt{1 + \lambda^2\epsilon^2\cos^2\lambda x}}[-\lambda\epsilon\cos\lambda x\psi_x + \psi_z]|_{1+\epsilon\sin\lambda x} \\ &= \frac{1}{\sqrt{1 + \lambda^2\epsilon^2\cos^2\lambda x}}\Big[\psi_{0z}|_1 + \epsilon\sin\lambda x\psi_{0zz}|_1 \\ &\quad + \frac{\epsilon^2}{2}\sin^2\lambda x\psi_{0zzz}|_1 + \epsilon\psi_{1z}|_1 \\ &\quad + \epsilon^2\sin\lambda x\psi_{1zz}|_1 + \epsilon^2\psi_{2z}|_1 - \epsilon\lambda\cos\lambda x\psi_{0x}|_1 \end{aligned}$$

$$- \epsilon^2\lambda \cos \lambda x \sin \lambda x \psi_{0xz}|_1 - \epsilon^2\lambda \cos \lambda x \psi_{1x}|_1 + \ldots \Big] \quad (28)$$

Similarly Eq. 21 gives

$$0 = \psi_0|_0 + \epsilon g_1\psi_{0z}|_0 + \epsilon^2 g_2\psi_{0z}|_0 + \frac{\epsilon^2 g_1^2}{2}\psi_{0zz}|_0 + \epsilon\psi_1|_0 + \epsilon^2 g_1\psi_{1z}|_0 + \epsilon^2\psi_2|_0 + \ldots \quad (29)$$

Eq. 22 are more difficult. A rotation of the stress tensors from (s, n) axes to (x, z) axes gives:

$$0 = T_{ns} = -\frac{f_x}{1 + f_x^2} T_{xx} + \frac{1 - f_x^2}{1 + f_x^2} T_{xz} + \frac{f_x}{1 + f_x^2} T_{zz} \quad (30)$$

$$-p_a = T_{nn} = \frac{f_x^2}{1 + f_x^2} T_{xx} - \frac{2f_x}{1 + f_x^2} T_{xz} + \frac{1}{1 + f_x^2} T_{zz} + \frac{\Lambda f_{xx}}{(1 + f_x^2)^{3/2}} \quad (31)$$

But

$$T_{xx} = -p + 2u_x = -p + 2\psi_{xz}$$
$$T_{xz} = u_z - w_x = \psi_{zz} - \psi_{xx}$$
$$T_{zz} = -p + 2w_z = -p - 2\psi_{xz} \quad (32)$$

Using Eqs. 23-25 and 30-32, we find:

$$0 = -4(\epsilon g_{1x} + \epsilon^2 g_{2x})(\psi_{0xz} + \epsilon\psi_{1xz})|_{\epsilon g_1 + \epsilon^2 g_2} + (1 - \epsilon^2 g_{1x}^2)(\psi_{0zz} - \psi_{0xx} + \epsilon\psi_{1zz} - \epsilon\psi_{1xx} + \epsilon^2\psi_{2zz} - \epsilon^2\psi_{2xx})|_{\epsilon g_1 + \epsilon^2 g_2} + 0(\epsilon^3) \quad (33)$$

$$\frac{1}{2}(1 + \epsilon^2 g_{1x}^2)(p_0 + \epsilon p_1 + \epsilon^2 p_2 - p_a)|_{\epsilon g_1 + \epsilon^2 g_2} - \frac{\Lambda}{2}(\epsilon g_{1xx} + \epsilon^2 g_{2xx})$$
$$= -(1 - \epsilon^2 g_{1x}^2)(\psi_{0xz} + \epsilon\psi_{1xz} + \epsilon^2\psi_{2xz})|_{\epsilon g_1 + \epsilon^2 g_2} - (\epsilon g_{1x} + \epsilon^2 g_{2x})(\psi_{0zz} - \psi_{0xx} + \epsilon\psi_{1zz} - \epsilon\psi_{1xx})|_{\epsilon g_1 + \epsilon^2 g_2} + 0(\epsilon^3) \quad (34)$$

Thus, equating $0(\epsilon^0)$ terms, Eqs. 17, 27-29, 33 and 34 give:

$$\nabla^4\psi_0 = 0 \quad (35)$$

$$1 = \psi_0|_1, \quad 0 = \psi_{0z}|_1, \quad (36)$$

$$0 = \psi_0|_0, \quad 0 = \psi_{0zz}|_0 - \psi_{0xx}|_0 \quad (37)$$

$$\frac{1}{2}(p_0 - p_a)|_0 = -\psi_{0xz}|_0 \quad (38)$$

The zeroeth order solution is the flow down an inclined flat plate with a parabolic velocity profile:

$$\psi_0 = \frac{1}{2}(3z - z^3), \quad (39)$$

Using Eq. 39, Eqs. 12, 14, 16, 23, 24, and 26 give:

$$p_{0x} + \epsilon p_{1x} + \epsilon^2 p_{2x} = \psi_{0zzz} + \epsilon\psi_{1xxz} + \epsilon\psi_{1zzz} + \epsilon^2\psi_{2xxz} + \epsilon^2\psi_{2zzz} + \frac{\rho g \sin\alpha \cos\beta a^3}{\mu(Q_0 + \epsilon Q_1 + \epsilon^2 Q_2)} - \epsilon^2\kappa(\psi_{0z}\psi_{1xz} - \psi_{1x}\psi_{0zz}) + 0(\epsilon^3) \quad (40)$$

$$p_{0z} + \epsilon p_{1z} + \epsilon^2 p_{2z} = -\epsilon\psi_{1xxx} - \epsilon\psi_{1zzx} - \epsilon^2\psi_{2xxx} - \epsilon^2\psi_{2zzx} + \frac{\rho g \cos\alpha a^3}{\mu(Q_0 + \epsilon Q_1 + \epsilon^2 Q_2)} + \epsilon^2\kappa\psi_{0z}\psi_{1xx} + 0(\epsilon^3) \quad (41)$$

The ϵ^0 terms yield:

$$p_{0x} = -3 + \rho g \sin\alpha \cos\beta a^3/(\mu Q_0) \quad (42)$$

Since we do not expect any mean pressure gradient, p_{0x} cannot admit non-zero constants. This determines the primary flow rate Q_0:

$$Q_0 = \rho g \sin\alpha \cos\beta a^3/(3\nu) \quad (43)$$

From Eqs. 38 and 41, we also find:

$$p_0 = 3Bz + p_a \quad (44)$$

where $B \equiv \cot\alpha \sec\beta$. The first order equations are:

$$\nabla^4\psi_1 = \kappa(\psi_{0z}\nabla^2\psi_{0x} - \psi_{0x}\nabla^2\psi_{0z}) = 0 \quad (45)$$

$$0 = \sin\lambda x\psi_{0z}|_1 + \psi_1|_1,$$
$$0 = \sin\lambda x\psi_{0zz}|_1 + \psi_{1z}|_1 - \lambda\cos\lambda x\psi_{0x}|_1 \quad (46)$$

$$0 = g_1\psi_{0z}|_0 + \psi_1|_0, \quad (47)$$

$$0 = -4g_{1x}\psi_{0xz}|_0 + \psi_{1zz}|_0 - \psi_{1xx}|_0 + g_1(\psi_{0zzz} - \psi_{0xxz})|_0 \quad (48)$$

$$-\frac{\Lambda}{2}g_{1xx} + \frac{1}{2}(p_1|_0 + g_1 p_{0z}|_0) = -(\psi_{1xz} + g_1\psi_{0xzz})|_0 - g_{1x}(\psi_{0zz} - \psi_{0xx})|_0 \quad (49)$$

The boundary conditions indicate ψ_1 is periodic in x. From Eq. 40, we obtain:

$$p_{1x} = \psi_{1xxz} + \psi_{1zzz} - \frac{\rho g \sin\alpha \cos\beta a^3}{\mu Q_0^2} Q_1 \quad (50)$$

Since p_{1x} cannot admit non-zero constants and ψ_1 is periodic,

$$p_{1x}|_0 = (\psi_{1xxz} + \psi_{1zzz})|_0, \quad Q_1 = 0 \quad (51)$$

We differentiate Eq. 49 with respect to x and eliminate p_1 from Eq. 51.

$$\frac{1}{2}[(\psi_{1xxz} + \psi_{1zzz})|_0 + g_{1x}3B] = -\psi_{1xxz}|_0 + \frac{\Lambda}{2}g_{1xxx} \quad (52)$$

The function g_1 is then eliminated from Eqs. 47 and 52.

$$\frac{1}{2}[(\psi_{1xxz} + \psi_{1zzz})|_0 - 2B\psi_{1x}|_0] = -\psi_{1xxz}|_0 - \frac{\Lambda}{3}\psi_{0xxx}|_0 \quad (53)$$

Similarly Eqs. 47 and 48 give:

$$\psi_{1zz}|_0 - \psi_{1xx}|_0 + 2\psi_1|_0 = 0 \quad (54)$$

Let

$$\psi_1 = e^{i\lambda x}F(z) \quad (55)$$

where the real part of the product is implied. Eqs. 45, 46, 53, and 54 give:

$$F'''' - 2\lambda^2 F'' + \lambda^4 F = 0 \quad (56)$$

$$F(1) = 0, \quad F'(1) = -3i \quad (57)$$

$$F'''(0) - 3\lambda^2 F'(0) - 2i\lambda\left(B + \frac{1}{3}\Lambda\lambda^2\right)F(0) = 0 \quad (58)$$

$$F''(0) + (\lambda^2 + 2)F(0) = 0 \quad (59)$$

The solution is:

$$F(z) = C\{-L\sinh\lambda z + \cosh\lambda z - Mz\sinh\lambda z + [(M + L)\tanh\lambda - 1]z\cosh\lambda z\} \quad (60)$$

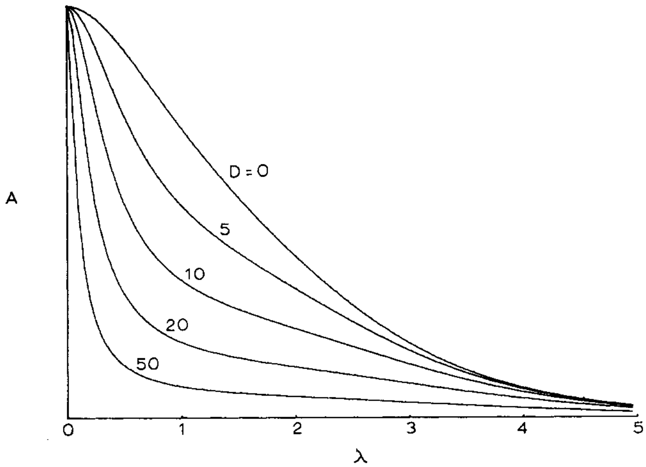

Figure 2. Amplitude of free surface.

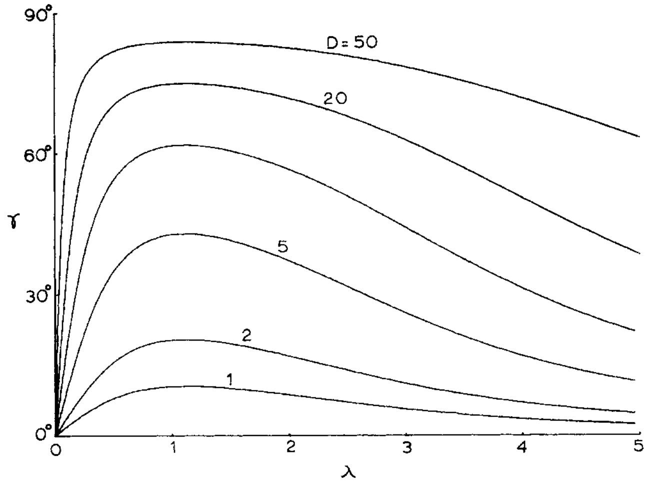

Figure 3. Phase shift of free surface.

where

$$C \equiv \frac{-3i}{L \sinh \lambda - \cosh \lambda(2 + \lambda^2 + \lambda L) + \lambda(M + L) \tanh \lambda \sinh \lambda} \tag{61}$$

and

$$L \equiv \frac{i}{\lambda^2} D, \quad M \equiv \lambda + \frac{1}{\lambda}, \quad D \equiv B + \frac{2}{3} \Lambda\lambda^2$$

We see that the effect of inclination angle and surface tension can be combined into a single parameter D. From Eqs. 47, 50, 55, and 60, we obtain the pressure and the shape of the free surface

$$p_1 = [i\lambda F'(z) - \frac{i}{\lambda} F'''(z)]e^{i\lambda x} \tag{62}$$

$$g_1 = -\frac{2}{3} \psi_1|_0 = -\frac{2}{3} e^{i\lambda x} C \equiv A \sin (\lambda x + \gamma) \tag{63}$$

The free surface has the same period as the wavy plate. However, its amplitude A and phase γ are complicated functions of D and λ. Figure 2 shows the amplitude of the free surface A. It is seen that the amplitude increases with decreasing λ (increasing wave length of the wavy plate) and with decreasing D (steeper plate angle with flow normal to direction of striations or less surface tension). Figure 3 shows the dependence of phase shift γ on λ and D. The free surface is in phase with the plate when $\lambda = 0$, $\lambda \to \infty$ or $D = 0$. Maximum phase shift of $\pi/2$ is attained as $\lambda \to 0$ and $D \to \infty$.

The second correction to the free surface, $g_2(x)$, may be obtained by simultaneously solving for $\psi_2(x, z)$. In what follows we shall concentrate only on the mean effect of the striations, which enters in the second order.

The Effect on Flow Rate

Since Q_1 is zero, any change in flow rate must be of second order. The problem is greatly simplified if we concentrate on the nonperiodic terms only. Let the nonperiodic part be denoted by a bar. Eq. 17 becomes:

$$\bar{\psi}_2''''(z) = 0 \tag{64}$$

The nonperiodic, $0(\epsilon^2)$ terms of Eqs. 27, 28, 29, 33, 34, 40 and 41 are:

$$0 = \frac{-3}{4} - \frac{1}{2i} F'(1) + \bar{\psi}_2(1) \tag{65}$$

$$0 = \frac{-3}{4} - \frac{1}{2i} F''(1) + \bar{\psi}_2'(1) - \frac{i\lambda^2}{2} F(1) \tag{66}$$

$$0 = \frac{3}{2} \bar{g}_2 - \frac{C^*}{3} F'(0) + \bar{\psi}_2(0) \tag{67}$$

$$0 = \frac{4}{3} \lambda^2 C^* F'(0) - 3\bar{g}_2 - \frac{C^*}{3} [F'''(0) + \lambda^2 F'(0)] + \bar{\psi}_2''(0) \tag{68}$$

$$\frac{1}{2} \left\{3B\bar{g}_2 - \frac{C^*}{3} [i\lambda^3 F(0) - i\lambda F''(0)] + \bar{p}_2|_0 + \frac{2}{9} \lambda^2 C C^* p_a\right\} = -\frac{i\lambda^3 C^*}{3} F(0) \tag{69}$$

$$0 = \bar{\psi}_2'''(z) - 3 \frac{\bar{Q}_2}{Q_0} \tag{70}$$

$$\bar{p}_{2z} = -3B \frac{\bar{Q}_2}{Q_0} \tag{71}$$

Here C^* is the complex conjugate of C and we have made use of the condition that $g_2(x)$ have zero mean. Eqs. 63-68 uniquely determine $\bar{\psi}_2$:

$$\bar{\psi}_2 = A_0 + A_1 z + A_2 z^2 + A_3 z^3 \tag{72}$$

The increase in flow is obtained from Eq. 70:

$$\begin{aligned}\frac{\bar{Q}_2}{Q_0} &= \frac{1}{3} \bar{\psi}_2''' = 2A_3 = \bar{\psi}_2(0) - \frac{1}{2} \bar{\psi}_2''(0) + \bar{\psi}_2'(1) - \bar{\psi}_2(1) \\ &= \frac{CC^*}{3} (M \tanh \lambda - 1) + \frac{3}{2} - iC[\lambda(M + L) \tanh \lambda \sinh \lambda - M\lambda \cosh \lambda - \lambda \sinh \lambda]\end{aligned} \tag{73}$$

Figure 4 shows $\bar{Q}_2/Q_0$ as a function of λ and D. Since $\bar{Q}_2$ is negative, a decrease in mean flow rate due to the waviness of the plate is observed. The decrease is greatest for large D and large λ. As $\lambda \to \infty$, $\bar{Q}_2/Q_0 \to 3/2 - 3\lambda$.

Flow Along Striations

Having determined the location of the free surface, we are

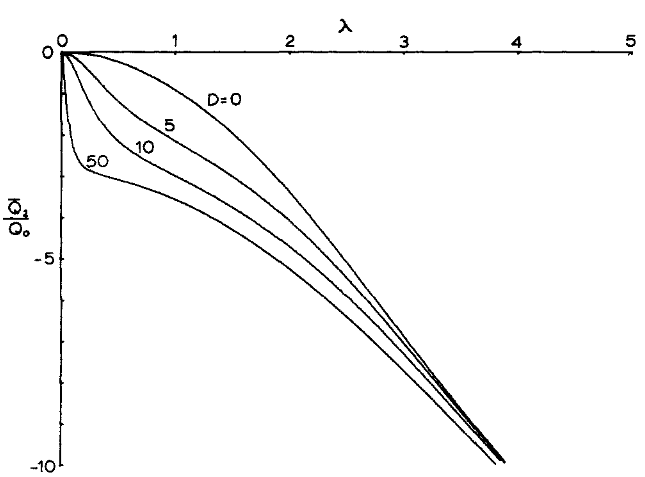

Figure 4. Increase in mean flow in x direction.

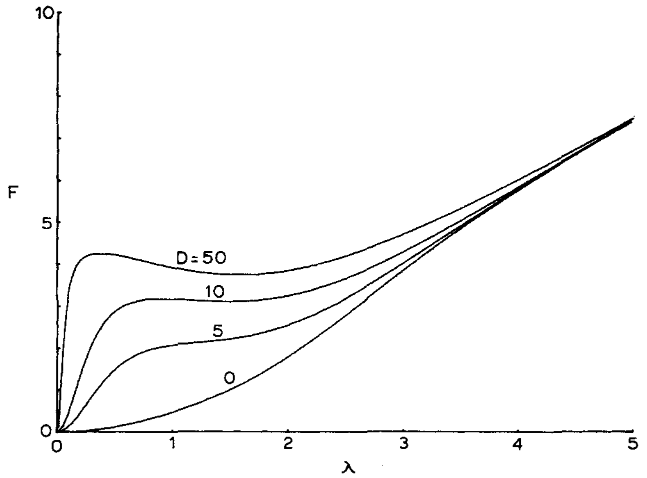

Figure 5. Increase in mean flow in y direction.

now in the position to obtain the flow in the $\underset{\sim}{j}$ direction. The boundary conditions are:

$$\text{on} \quad z = 1 + \epsilon \sin \lambda x, \quad v = 0 \tag{74}$$

$$\text{on } z = \epsilon g_1 + \epsilon^2 g_2 + \ldots,$$

$$T_{ny} = 0 \text{ or } -(\epsilon g_{1_x} + \epsilon^2 g_{2_x} + \ldots)\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z} = 0 \tag{75}$$

We expand

$$v = v_0 + \epsilon v_1 + \epsilon^2 v_2 + \ldots \tag{76}$$

Eqs. 13, 74, and 75 yield:

$$v_{0xx} + v_{0zz} = -\frac{g \sin\alpha \sin\beta a^3}{\nu Q_0} = -3\tan\beta \tag{77}$$

$$v_0|_1 = 0, \quad \frac{\partial v_0}{\partial z}|_0 = 0 \tag{78}$$

From which we obtain:

$$v_0 = \frac{3\tan\beta}{2}(1 - z^2) \tag{79}$$

The next order equations are:

$$v_{1xx} + v_{1zz} = 3\tan\beta\frac{Q_1}{Q_0} = 0 \tag{80}$$

$$v_1|_1 = -\sin\lambda x v_{0z}|_1 = 3\sin\lambda x \tan\beta = -3i\tan\beta e^{i\lambda x} \tag{81}$$

$$v_{1z}|_0 = g_{1_x} v_{0_x}|_0 - g_1 v_{0zz}|_0 = -2\tan\beta C e^{i\lambda x} \tag{82}$$

The solution is:

$$v_1 = 3\tan\beta e^{i\lambda x}\Big[\Big(\frac{2C}{3\lambda}\tanh\lambda - i\,\text{sech}\,\lambda\Big)\cosh\lambda z - \frac{2C}{3\lambda}\sinh\lambda z\Big] \tag{83}$$

Since v_1 is periodic in x, any contribution to net flow rate in the $\underset{\sim}{j}$ direction is of order ϵ^2. We have

$$\bar{v}_{2zz} = 3\tan\beta\frac{\overline{Q}_2}{Q_0} \tag{84}$$

$$\bar{v}_2|_1 = -\overline{\sin\lambda x v_{1z}|_1} - \overline{\frac{\sin^2\lambda x}{2}v_{0zz}|_1}$$
$$= \tan\beta\Big(iC\,\text{sech}\,\lambda - \frac{3}{2}\lambda\tanh\lambda + \frac{3}{4}\Big) \tag{85}$$

$$\bar{v}_{2z}|_0 = \overline{g_{1_x}v_{1_x}|_0} - \overline{g_2 v_{0zz}|_0} - \overline{g_1 v_{1zz}|_0} = 0 \tag{86}$$

The solution is:

$$\bar{v}_2 = \tan\beta\Big[\frac{3\overline{Q}_2}{2Q_0}(z^2 - 1) + iC\,\text{sech}\,\lambda - \frac{3}{2}\lambda\tanh\lambda + \frac{3}{4}\Big] \tag{87}$$

The total mean flow in $\underset{\sim}{j}$ direction, per unit width, is:

$$\overline{\int_{\epsilon g_1+\epsilon^2 g_2}^{1+\epsilon\sin\lambda x} v dz} = \int_0^1 \bar{v}\, dz + \epsilon^2\,\overline{\sin\lambda x v_1|_1} - \epsilon^2\,\overline{g_1 v_1|_0} + \frac{\epsilon^2}{2}\overline{\sin^2\lambda x v_{0z}|_1} - \frac{\epsilon^2}{2}\overline{g_1^2 v_{0z}|_0} + 0(\epsilon^4)$$
$$= \tan\beta + \epsilon^2\tan\beta\Big\{-\frac{CC^*}{3}\Big[\Big(M - \frac{2}{\lambda}\Big)\tanh\lambda - 1\Big] - \frac{3}{2}\lambda\tanh\lambda + iC[\lambda(M+L)\tanh\lambda\sinh\lambda - M\lambda\cosh\lambda - \lambda\sinh\lambda + 2\,\text{sech}\,\lambda]\Big\} + 0(\epsilon^4)$$
$$\equiv \tan\beta[1 + \epsilon^2 F(\lambda, D) + 0(\epsilon^4)] \tag{88}$$

We see that the flow in the $\underset{\sim}{j}$ direction is increased by $\epsilon^2\tan\beta F(\lambda, D)$ due to the waviness of the bottom plate. The function $F(\lambda, D)$ is plotted in Figure 5. Also, the flow in $\underset{\sim}{j}$ direction is increased by increased D. For fixed D, increased λ in general increases flow, inspite of the existence of a local maximum for $D \geq 7.21$, $\lambda \leq 1.20$. The asymptote for large λ is $F \sim 3\lambda/2$. The decrease of flow in the x direction is not compensated by the increase in flow in the y direction.

Drift

We shall discuss the "drift" of a particle on the free surface. Let $\underset{\sim}{r}$ be the direction of the velocity of a particle on the free surface if the bottom plate were flat, and let $\underset{\sim}{V}$ be its planar velocity vector if the plate is wavy. More precisely,

$$\underset{\sim}{r} \equiv \underset{\sim}{g} - (\underset{\sim}{g}\cdot\underset{\sim}{k})\underset{\sim}{k} \tag{89}$$

$$\underset{\sim}{V} \equiv u\underset{\sim}{i} + v\underset{\sim}{j} \tag{90}$$

Figure 1 shows $\underset{\sim}{V}$ and $\underset{\sim}{r}$ differ by a drift angle δ. We find

$$\tan(\beta + \delta) = \Big(\frac{v_0 + \epsilon v_1 + \epsilon^2 v_2}{u_0 + \epsilon u_1 + \epsilon^2 u_2}\Big)\Big|_{\epsilon g_1 + \epsilon^2 g_2} + \ldots \tag{91}$$

To the first order (Eq. 91) gives:

$$\delta = \epsilon\cos\beta\sin\beta\Big(\frac{v_1}{v_0} - \frac{u_1}{u_0}\Big)\Big|_0 + 0(\epsilon^2)$$

$$= \epsilon \cos \beta \sin \beta 2 e^{i\lambda x}\left\{\frac{2C}{3\lambda} \tanh \lambda - i \operatorname{sech} \lambda - \frac{C}{3}[(M + L) \tanh \lambda - L\lambda - 1]\right\} + 0(\epsilon^2)$$
$$\equiv \epsilon H(\lambda, \beta, D) e^{i(\lambda x + \eta)} + 0(\epsilon^2) \quad (92)$$

where H is real. The *mean* drift angle is of second order:

$$\bar{\delta} = \epsilon^2 \sin \beta \cos \beta \left[\frac{\overline{v_2}}{v_0} - \frac{\overline{u_1 v_1}}{u_0 v_0} - \frac{\overline{u_2}}{u_0} + \frac{\overline{u_1^2}}{u_0^2} + \frac{\partial}{\partial z}\left(\frac{\overline{g_1 v_1}}{v_0} - \frac{\overline{g_1 u_1}}{u_0} \right) \right]_0 + 0(\epsilon^4)$$
$$= \epsilon^2 \sin \beta \cos \beta S(\lambda, D) + 0(\epsilon^4) \quad (93)$$

$S(\lambda, D)$ can be obtained from the velocities found previously. We find the mean drift angle increases with the square of the amplitude of bottom corrugations. It is also maximum when $\beta = 45°$ and when λ and D are large.

DISCUSSION

We mention here the related stability problem of film flow down a smooth incline which also involves a wavy free surface. The major physical difference from the present paper is that the surface waves of the stability problem are unsteady, and may occur only at higher Reynolds numbers while the surface waves caused solely by bottom striations are steady, completely deterministic and occur for any Reynolds number (although only the small Reynolds number case is analyzed here). At high Reynolds numbers the bottom striations may excite unstable waves of equal wave length, which is a destabilizing effect.

Mathematically, our first order correction is similar to the stability problem, albeit with different boundary conditions. The infinitesimal stability problem was studied by Benjamin (1957) and Yih (1963). The linearized Orr-Somerfeld equation was solved but the wave amplitude cannot be determined. The finite amplitude stability problem was investigated by Benny (1966), Lin (1969) and Krantz and Goren (1970), who studied the nonlinear growth of small amplitude waves. None of the previous authors studied the second order affects which govern the changes in flow rate and drift.

The fluid flow of *infinite* depth over a wavy bottom was studied by Benjamin (1959), Thorsness et al. (1978). Again, the theory is of first order without the second order effects studied here. The present paper is further complicated by the existence of a free surface which characterizes film flow and a three dimensionality caused by striations at an angle with the horizontal.

NOTATION

a = depth of fluid
A = amplitude of g_1
b = amplitude of wavy plate
B = $\cot \alpha \sec \beta = \underset{\sim}{g} \cdot \underset{\sim}{k} / \underset{\sim}{g} \cdot \underset{\sim}{i}$
C = a function of λ, D
D = $B + \frac{1}{3} \Lambda \lambda^2$
f = function of x
F = a function of λ, D
$\underset{\sim}{g}$ = gravitational acceleration
H = function of λ, β, D
$\underset{\sim}{i}$ = unit vector
$\underset{\sim}{j}$ = unit vector
$\underset{\sim}{k}$ = unit vector
l = period of wavy plate
L = iD/λ^2
M = $\lambda + l/\lambda$
n = direction normal to boundary
p = pressure
p_a = ambient pressure
Q = flow rate
$\underset{\sim}{r}$ = $\underset{\sim}{g} - (\underset{\sim}{g} \cdot \underset{\sim}{k})\underset{\sim}{k}$
R = Reynolds number = Q/ν
s = direction tangent to boundary
S = function of λ, D
t = surface tension
T_{ns} = normalized stress tensor
T_{nn} = normalized stress tensor
u = velocity in x direction
v = velocity in y direction
$\underset{\sim}{V}$ = planar velocity $u\underset{\sim}{w} + v\underset{\sim}{g}$
w = velocity in z direction
x = Cartesian coordinate
y = Cartesian coordinate
z = Cartesian coordinate

Greek letters

α = mean angle of incline of the plate
β = angle of the wavy striations
δ = drift angle
ϵ = b/a
η = phase shift of drift
κ = Ra/b
λ = $2\pi a/l$
Λ = normalized surface tension $ta/(\rho\nu Q)$
ν = kinematic viscosity
ρ = density
ψ = stream function
γ = phase shift of g_1

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